

Deformed boson algebras and the quantum double construction

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Abstract

The quantum double construction of a q -deformed boson algebra possessing a Hopf algebra structure is carried out explicitly. The R -matrix thus obtained is compared with the existing literature.

Recently there has been an increasing interest in the deformation of Lie (super)algebras[1, 2, 3, 4, 5, 6] and their quasitriangular Hopf algebra nature[7], mainly because of there wide applications in mathematical physics. Parallel attempts to consistently q -deform the boson algebra also appeared[8, 9, 10, 11, 12, 13, 14, 15, 16] both independently and in connection with quantum group realizations, addressing also their possible Hopf algebra nature[17, 18]. The main aim of this letter is, on the one hand to point out the ambiguous validity of an R -matrix obtained from a definition of q -boson algebra endowed with a Hopf algebra structure[17], and on the other to demonstrate the quantum double construction[1, 19, 20] for this algebra which will lead to an unambiguously valid R -matrix.

The q -deformed boson algebras, denoted here by L , that have been considered are usually taken to be generated by a , a^\dagger and N subject to the following commutation relations:

$$\begin{aligned} [N, a] &= -a, \\ [N, a^\dagger] &= a^\dagger, \end{aligned} \tag{1}$$

together with *one* out of the following list of additional relations:

$$[a, a^\dagger] = [N + I]_q - [N]_q, \tag{2}$$

$$aa^\dagger - q^{-1}a^\dagger a = q^N, \tag{3}$$

$$aa^\dagger - qa^\dagger a = q^{-N}, \tag{4}$$

$$a^\dagger a = [N], \quad \text{and} \quad aa^\dagger = [N + I], \tag{5}$$

where I is the unit of L and as usual $[x] = (q^x - q^{-x})/(q - q^{-1})$, and q not a root of unity. When $q = 1$ we obtain the well known defining relations of the undeformed boson algebra. It should be mentioned that the consistency of the above definitions is justified as they can also be obtained from $sl_q(2)$ by contraction[13, 14, 21]. Generalizations of q -boson defining relations, in particular that of (3), (4) have also been studied[9, 22, 23, 24]. Analysis of representations of L is quite rich [25, 24], but the most ususally used is the q -Fock representation (which has been shown [27] to be isomorphic with the usual boson Fock space by expressing the q -bosons as suitable functions of the undeformed bosons) given by:

$$\begin{aligned} |n\rangle &= ([n]!)^{1/2} (a^\dagger)^n |0\rangle, \quad N|n\rangle = n|n\rangle, \\ a^\dagger |n\rangle &= [n+1]^{1/2} |n+1\rangle, \quad a|n\rangle = [n]^{1/2} |n-1\rangle \end{aligned} \tag{6}$$

where $n = 0, 1, \dots$. Using this representation, one can also show [21, 27, 18] the equivalence amongst the above definitions, which *does not* imply, though, an equivalence at the abstract algebraic level (as has been demonstrated in [18]).

The most important point though concerns the Hopf algebra structure of the deformed boson algebra. Initially Hong Yan[17] showed that when L is defined by (1) and (2) (with $N \rightarrow N - 1/2$, see (7) below) L is a Hopf algebra. Later this result was generalized in [18] where (1) was also generalized. We shall concentrate hereafter on the Hopf algebra L as defined in [17] by (1) and a symmetrized version of (2), namely

$$[a, a^\dagger] = \left[N + \frac{1}{2}\right]_q - \left[N - \frac{1}{2}\right]_q. \tag{7}$$

The structure of this letter is as follows. First we give general information on quasitriangular Hopf algebras, and also present the model of [17], focussing mainly on the claimed R -matrix and pointing out some inconsistencies in its properties. Then, by demonstrating the method of quantum double construction, we apply it to the Hopf algebra L defined by (1) and (7) to obtain a valid R -matrix which can be compared with that of [17].

Consider an algebra A , say over \mathbb{C} , with multiplication $m : A \otimes A \rightarrow A$ (i.e. $m(a \otimes b) = ab$, $\forall a$ and $\forall b \in A$) and unit $u : \mathbb{C} \rightarrow A$ (i.e. $u(1) = I$, the identity on A) endowed with a Hopf algebra structure that is, having a coproduct $\Delta : A \rightarrow A \otimes A$, counit $\epsilon : A \rightarrow \mathbb{C}$ (which is a homomorphism) and antipode $S : A \rightarrow A$ (which is an antihomomorphism i.e. $S(ab) = S(b)S(a)$) with the following consistency conditions:

$$\begin{aligned} (id \otimes \Delta)\Delta(a) &= (\Delta \otimes id)\Delta(a) \\ m(id \otimes S)\Delta(a) &= m(S \otimes id)\Delta(a) = \epsilon(a)1, \\ (\epsilon \otimes id)\Delta(a) &= (id \otimes \epsilon)\Delta(a) = a \end{aligned} \quad (8)$$

Note also that we have $\epsilon(I) = 1$, $S(I) = I$ and $\epsilon(S(a)) = \epsilon(a)$, $\forall a \in A$. Following Sweedler [7] we write:

$$\begin{aligned} \Delta(a) &= \sum_{(a)} a^{(1)} \otimes a^{(2)} \quad \forall a \in A \quad \text{and generally,} \\ \Delta_n(a) &= (\Delta \otimes I^{\otimes(n-1)})\Delta_{n-1}(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \dots \otimes a^{(n+1)} \quad \text{for } n \geq 2 \end{aligned} \quad (9)$$

Let T denote the twist map on $A \otimes A$, $T(a \otimes b) = b \otimes a$ and assume that S^{-1} , the inverse of the antipode, exists. Then there exists an opposite Hopf algebra structure on A with coproduct and antipode $T\Delta$ and S^{-1} respectively. According to Drinfeld [1] a Hopf algebra A is called quasitriangular if there exists an invertible element R such that

$$\begin{aligned} R &= \sum_i a_i \otimes b_i \in A \otimes A \\ T\Delta(a)R &= R\Delta(a) \quad \forall a \in A \end{aligned} \quad (10)$$

Then it can be shown that R satisfies the Yang-Baxter equation

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12} \quad \text{where} \\ (\Delta \otimes I)R &= R_{13}R_{23}, \quad (I \otimes \Delta)R = R_{13}R_{12}. \end{aligned} \quad (11)$$

Turning now to the algebra L given by (1), (7), the coproduct Δ , counit ϵ and antipode S are respectively as follows:

$$\begin{aligned} \Delta(N) &= N \otimes I + I \otimes N - \frac{i\alpha}{\gamma} I \otimes I, \\ \Delta(a) &= \left(a \otimes q^{N/2} + iq^{-N/2} \otimes a \right) e^{-i\alpha/2}, \\ \Delta(a^\dagger) &= \left(a^\dagger \otimes q^{N/2} + iq^{-N/2} \otimes a^\dagger \right) e^{-i\alpha/2}, \\ \epsilon(N) &= \frac{i\alpha}{\gamma}, \quad \epsilon(a) = \epsilon(a^\dagger) = 0, \quad \epsilon(I) = 1, \\ S(N) &= -N + \frac{2i\alpha}{\gamma} I, \\ S(a) &= -q^{-1/2} a, \\ S(a^\dagger) &= -q^{1/2} a^\dagger. \end{aligned} \quad (12)$$

where $\alpha = 2\kappa\pi + \pi/2, (\kappa \in \mathbb{Z})$ and $\gamma = \ln q$. The consistency of these operations can be verified by direct calculation using (1), (7) and the consistency relations (8). Moreover the ideal K of L generated by $C = a^\dagger a - [N - \frac{1}{2}]_q$ is not a Hopf ideal (as $\Delta(C) \notin K \otimes L + L \otimes K$) and thus the quotient algebra isomorphic to the one generated by (1) and both $a^\dagger a = [N - \frac{1}{2}]_q$, $aa^\dagger = [N + \frac{1}{2}]_q$, is not a Hopf algebra.

Although the Fock space (6), but with $N|n\rangle = (n + 1/2)|n\rangle$, furnishes a representation of L , we can use a more general one given by:

$$\begin{aligned} N|n\rangle &= (n + c)|n\rangle \\ a|n\rangle &= ([n + c - 1/2] - [c - 1/2])^{1/2}|n - 1\rangle \\ a^\dagger|n\rangle &= ([n + c + 1/2] - [c - 1/2])^{1/2}|n + 1\rangle \end{aligned} \quad (13)$$

where c is a non-zero complex number. If $c = \frac{1}{2}$ we obtain the space just mentioned above. For $|q| \neq 1$ we can choose the states $|n\rangle$ to be given by

$$\begin{aligned} |n\rangle &= \left(\frac{i^n (q^{1/2} + q^{-1/2})^n \Gamma_{q^{1/2}}^+ \left(2c + \frac{i\pi}{2\ln q} \right)}{(q^{1/2} - q^{-1/2})^n [n]_{q^{1/2}}! \Gamma_{q^{1/2}}^+ \left(n + 2c + \frac{i\pi}{2\ln q} \right)} \right)^{1/2} a^{\dagger n} |0\rangle, \quad \text{or} \\ |n\rangle &= \left(\frac{(-i)^n (q^{1/2} + q^{-1/2})^n \Gamma_{q^{1/2}}^+ \left(-n - 2c - \frac{i\pi}{2\ln q} \right)}{(q^{1/2} - q^{-1/2})^n [n]_{q^{1/2}}! \Gamma_{q^{1/2}}^+ \left(-2c - \frac{i\pi}{2\ln q} \right)} \right)^{1/2} a^{\dagger n} |0\rangle, \end{aligned} \quad (14)$$

whichever is well defined given a fixed value of c and $\ln q$. Note that at least one of the above expressions is always well-defined, and if they are both well-defined, they are equal. The symmetric $q^{1/2}$ -factorial $[n]_{q^{1/2}}!$ is defined similarly to the symmetric q -factorial as in [10, 28], and the symmetric $q^{1/2}$ -gamma function $\Gamma_{q^{1/2}}^+(z)$ similarly to the symmetric q -gamma function $\Gamma_q^+(z)$ as in [29] and $\langle 0|0\rangle = 1$. If we now consider the quantum algebra $U_{q^{1/2}}(sl(2))$ defined by:

$$[e, f] = [h]_{q^{1/2}}, \quad [h, e] = 2e, \quad [h, f] = -2f, \quad (15)$$

we can easily verify that the following expressions of h, e, f in terms of the q -bosons do indeed satisfy (15):

$$h = 2N - \frac{2i\alpha}{\gamma}, \quad e = \lambda a^\dagger, \quad f = \frac{i(q^{1/2} + q^{-1/2})}{\lambda(q^{1/2} - q^{-1/2})} a, \quad (16)$$

where λ is some constant. In this realization of $U_{q^{1/2}}(sl(2))$, the central element C is the deformed quadratic Casimir. Conversely, given $U_q(sl(2))$ as in (15) (with $q^{1/2} \rightarrow q$) and defining

$$N = \frac{1}{2}h + \frac{i\alpha}{4\ln q}, \quad a = \mu f, \quad a^\dagger = \frac{-i(q - q^{-1})}{\mu(q + q^{-1})} e, \quad (17)$$

where μ is some constant, then N, a, a^\dagger satisfy the relations (7) with q^2 in the place of q , and so the q^2 -boson algebra is isomorphic to a quotient algebra of $U_q(sl(2))$. This realization is similar to that of [26]. Finally it should also be mentioned that a q -deformed differential operator algebra was associated with L also possessing a Hopf algebra structure in [16].

The author of [17] defines an R -matrix of L as the invertible element that intertwines between the coproduct of (12) and a coproduct $\bar{\Delta}$ that is obtained from that of (12) by changing $q \rightarrow q^{-1}$ so that

$$R\Delta = \bar{\Delta}R \quad (18)$$

This definition is claimed to lead to the following R -matrix[17]

$$R = q^{(N - \frac{i\alpha}{\gamma}I) \otimes (N - \frac{i\alpha}{\gamma}I) - \frac{1}{2}N \otimes N} \sum_{k=0}^{\infty} \frac{i^k (1 + q^{-1})^k q^{-k(k+1)/4}}{\prod_{j=1}^k \left[\frac{j}{2} \right]_q} (a^\dagger)^k \otimes q^{-kN/2} a^k. \quad (19)$$

Moreover the author states that R satisfies the Yang–Baxter equation (11) (the same R appears also in [26]). Certain comments relative to the above definition of R have to be made. One would expect that R should be defined by relation such as (10) as this would justify its nature as an interwiner between the Hopf algebra structure and the opposite one. Instead it seems that in [17] R is not treated as such. $\bar{\Delta}$ together with the counit and antipode given in (12) does not constitute a Hopf algebra, not even a coalgebra, and also $\bar{\Delta}$ taken with the counit and the inverse of S in (12) does not constitute a Hopf algebra either. It is the Hopf structure obtained from (12) by setting $q \rightarrow q^{-1}$ everywhere which is consistent with $\bar{\Delta}$ (obviously this change leaves (7) unaffected). It should be noted that definition (18) is reminiscent of the case for quantum groups[5] where the definition of the reduced \bar{R} -matrix, is given by $\bar{R}T\bar{\Delta} = T\bar{\Delta}\bar{R}$ (T being the twist map) and satisfies relations similar to the Yang–Baxter equation (the R -matrix then can be expressed as a product of \bar{R} with an appropriate q -exponentiated function of the Cartan subalgebra basis of the Lie algebra and satisfies (10)). In fact \bar{R} has also been used by Lusztig[30]. These considerations suggest that (18) alone cannot be used as a definition either of an R -matrix or of a reduced \bar{R} -matrix. One can indeed verify by direct calculation that R given by (19) *does not* satisfy any of the relations (18), (11), (10) or the relation for the reduced \bar{R} -matrix mentioned above, for example $R\Delta(N) = \Delta(N)R$. It is the implementation of the natural definition of R (10), that leads to the correct R -matrix (where $T\Delta$ is compatible with the counit and coproduct given in (12)) that we shall demonstrate in what follows using the quantum double construction whose structure will now be presented.

Let A^* denote the dual of A with elements a^* defined by $(a^*, b) = a^*(b)$, $\forall a \in A^*$ and $\forall b \in A$, where $(\ , \)$ is the natural bilinear form $A^* \otimes A \rightarrow \mathbb{C}$ (with A and A^* regarded as vector spaces). We assume that

$$A^0 = [a^* \in A^* | kera^* \text{ contains a cofinite two sided ideal of } A]$$

is dense in A^* i.e. $(A^0)^\perp = [a \in A | (b^*, a) = 0, \ \forall b^* \in A^0] = (0)$. For A finite dimensional, $A^0 = A^*$. Moreover if A is such that the intersection of all cofinite two-sided ideals is (0) then for every $a \in A$ and every $b^* \in A^0$ we have

$$a = \sum_s a_s (a_s^*, a)$$

$$b^* = \sum_s (b^*, a_s) a_s^*$$

where a_s and a_s^* are the basis of A and A^0 such that $(a_s^*, a_t) = \delta_{st}$ and $s = 1, 2, \dots, \dim A$. Following Sweedler[7] we have:

Theorem: A^0 becomes a Hopf algebra with multiplication m^0 , unit u^0 , coproduct Δ^0 , antipode S^0 and counit ϵ^0 defined by:

$$m^0 = \Delta^*|_{A^0 \otimes A^0}, \quad u^0 = \epsilon^*|_{A^0} \quad (20)$$

$$\Delta^0 = m^*|_{A^0}, \quad S^0 = S^*|_{A^0}, \quad (21)$$

$$\epsilon^0(a^*) = (a^*, 1) \quad \forall a^* \in A^0 \quad (22)$$

where m^* , Δ^* , ϵ^* , and S^* are the dual maps of m , Δ , ϵ , S respectively. The identity element of A^0 is given by the counit of A .

In what will follow we shall consider the opposite Hopf algebra structure on A^0 , where we keep the same multiplication, unit and counit of A^0 , but we use the coproduct $T\Delta^0$ and antipode $(S^0)^{-1}$ given by

$$(T\Delta^0(a^*), b \otimes c) = (a^*, cb) \quad (23)$$

$$((S^0)^{-1}(a^*), b) = (a^*, S^{-1}(b)), \quad (24)$$

$\forall a^* \in A^0$ and $b, c \in A$. Finally it should be borne in mind that both $A \otimes A^0$ and $A^0 \otimes A$ inherit a Hopf algebra structure with respective coproducts Δ' and Δ'' given by

$$\Delta' = (I \otimes \tau \otimes I)(\Delta \otimes T\Delta^0), \quad (25)$$

$$\Delta'' = (I \otimes \tau^{-1} \otimes I)(T\Delta^0 \otimes \Delta), \quad (26)$$

where τ is the twist isomorphism $A \otimes A^0 \rightarrow A^0 \otimes A$ given by $\tau(a \otimes b^*) = b^* \otimes a$.

In the quantum double construction[1, 20] we first construct the vector space $D(A)$ called the quantum double of A which is the vector space of all free products of the form ab^* , $\forall a \in A$ and $\forall b^* \in A^0$. $D(A)$ is isomorphic (as a vector space) with $A \otimes A^0$, the isomorphism being given by $\psi(a \otimes b^*) = ab^*$, $\forall a \in A$ and $\forall b^* \in A^0$.

$D(A)$ becomes an algebra by defining all products of the form b^*a , $\forall a \in A$ and $\forall b^* \in A^0$, as $b^*a = \mu(b^* \otimes a)$ where the $\mu : A^0 \otimes A \rightarrow D(A)$ given by:

$$A^0 \otimes A \xrightarrow{(tr \otimes I \otimes 2)((S^0)^{-1} \otimes I \otimes 3)\Delta''} A^0 \otimes A \xrightarrow{(I \otimes 2 \otimes tr)\Delta''} A^0 \otimes A \xrightarrow{\tau^{-1}} A \otimes A^0 \xrightarrow{\psi} D(A) \quad (27)$$

where $tr : A^0 \otimes A \rightarrow \mathbb{C}$ is given by $tr(b^* \otimes a) = (b^*, a)$. Explicitly we have

$$b^*a = \sum_{(a), (b^*)} ((S^0)^{-1}((b^*)^{(1)}, a^{(1)})((b^*)^{(3)}, a^{(3)})a^{(2)}(b^*)^{(2)}) \quad (28)$$

Both A and A^0 are embedded in $D(A) = A \otimes A^0$ by identifying Ia^* and $a\epsilon$ with a^* and a respectively, $\forall a^* \in A^0$ and $\forall a \in A$.

$D(A)$ becomes a quasitriangular Hopf algebra with coproduct Δ_D , counit ϵ_D , and antipode S_D and canonical element R given by

$$\Delta_D(ab^*) = \Delta(a)(T\Delta^0)(b^*) \quad (29)$$

$$\epsilon_D(ab^*) = \epsilon(a)\epsilon^0(b^*) \quad (30)$$

$$S_D(ab^*) = (S^0)^{-1}(b^*)S(a) \quad (31)$$

$$R = \sum_s a_s \otimes a_s^* \in D(A) \otimes D(A) \quad (32)$$

$$R^{-1} = (S_D \otimes I)R \quad (33)$$

where $[a_s]$ and $[a_t^*]$ are bases of A and A^0 respectively such that $(a_s^*, a_t) = \delta_{st}$.

From now on we should always treat the Hopf algebra L , as belonging in the category of quantized universal enveloping algebras. In that sense we consider L to be spanned by elements of the form $(N - \frac{i\alpha}{\gamma}I)^m \cdot q^{-kN/2} a^k \cdot q^{lN/2} (a^\dagger)^l$ ($l, m, k = 0, 1, 2, \dots$). We shall denote by L_+ and L_- the Hopf subalgebras spanned by $(N - \frac{i\alpha}{\gamma}I)^m \cdot q^{-kN/2} a^k$ and $(N - \frac{i\alpha}{\gamma}I)^m \cdot q^{lN/2} (a^\dagger)^l$ respectively.

Following the method just described and similar in spirit with [19] we put $A = L_+$ and construct the quantum double $D(L_+)$ of L_+ , to obtained an R -matrix for L compatible with definitions (10) and (11). We shall denote by u and m the unit and multiplication of L_+ while the coproduct Δ , counit ϵ and antipode S are as in (12). Taking L_+ to be generated by $N - (i\alpha/\gamma)I$ and $q^{N/2}a^\dagger$, we shall denote its basis by $e_{km} = q^{kN/2} \left(N - \frac{i\alpha}{\gamma}I\right)^m (a^\dagger)^k$ with $k, m \geq 0$.

As L_+ is a coalgebra its dual L_+^* is necessarily an algebra and via the above theorem, L_+^0 becomes a Hopf algebra. Let us now construct explicitly L_+^0 . As a vector space L_+^0 will consist of all linear maps of L_+ on to \mathbb{C} . We take L_+^0 to be generated by the functionals ν, β on L_+ and taking values in \mathbb{C} defined by

$$\begin{aligned} \nu \left(\left(N - \frac{i\alpha}{\gamma}I \right)^n q^{\frac{lN}{2}} (a^\dagger)^l \right) &= \frac{\delta_{l0}\delta_{n1} + i\alpha\delta_{l0}\delta_{n0}}{\gamma}, \\ \beta \left(\left(N - \frac{i\alpha}{\gamma}I \right)^n q^{\frac{lN}{2}} (a^\dagger)^l \right) &= \frac{e^{-i\alpha/2}\delta_{l1}}{2^n(1+q^{-1})}, \end{aligned} \quad (34)$$

and extending by linearity.

The Hopf algebra structure of L_+^0 can easily be found from the above theorem. We are interested in the opposite Hopf algebra structure of L_+^0 where the multiplication, unit and counit are as in the theorem but we use the opposite coproduct $T\Delta^0$ and antipode $(S^0)^{-1}$ on L_+^0 given by :

$$T\Delta^0(\nu) = \nu \otimes 1^* + 1^* \otimes \nu - \frac{i\alpha}{\gamma} 1 \otimes 1^*, \quad (35)$$

$$T\Delta^0(\beta) = \left(\beta \otimes q^{\nu/2} + iq^{-\nu/2} \otimes \beta \right) e^{-i\alpha/2}, \quad (36)$$

$$(S^0)^{-1}(\nu) = -\nu + (2i\alpha/\gamma)1^*, \quad (37)$$

$$(S^0)^{-1}(\beta) = -\frac{1}{q^{1/2}}\beta \quad (38)$$

where 1^* is the identity on L_+^0 (i.e. $u^0(1) = 1^*$). Moreover as L_+^0 inherits a Lie algebra structure, with the non-zero Lie bracket given by

$$[\nu, \beta] = -\beta, \quad (39)$$

as can be seen using (35). It is convenient to define a basis of L_+^0 given by

$$e_{km}^* = \left(\nu - \frac{i\alpha}{\gamma}1^* \right)^m q^{-k\nu/2} \beta^k, \quad \text{so that} \quad (40)$$

$$\begin{aligned} e_{km}^*(e_{ln}) &= \left(q^{-k\nu/2} \left(\nu - \frac{i\alpha}{\gamma}1^* \right)^m \beta^k \right) \left(q^{lN/2} \left(N - \frac{i\alpha}{\gamma}I \right)^n (a^\dagger)^l \right) \\ &= \delta_{kl}\delta_{mn} \frac{n!(-i)^k q^{k(k+1)/4}}{\gamma^n} \cdot \prod_{j=1}^k \left[\frac{j}{2} \right]_q. \end{aligned} \quad (41)$$

Observe that the map $\nu \rightarrow N$ and $\beta \rightarrow a$ defines an isomorphism $L_+^0 \cong L_-$.

Considering now the quantum double $D(L_+) = L_+ \otimes L_+^0$. It is the vector space spanned by all free products ab^* which becomes an algebra by defining all products of the form b^*a as was indicated above. Moreover as was stated it is a quasitriangular Hopf algebra and thus by appropriately normalising the elements e_{km} and e_{lm}^* using (41), the canonical element R given in (32) is realised as

$$R = q^{(N - \frac{i\alpha}{\gamma}I) \otimes (\nu - \frac{i\alpha}{\gamma}1^*)} \sum_{k=0}^{\infty} \frac{i^k q^{-k(k+1)/4}}{\prod_{j=1}^k \left[\frac{j}{2} \right]_q} q^{kN/2} (a^\dagger)^k \otimes q^{-k\nu/2} \beta^k \in D(L_+). \quad (42)$$

The relation of our q -boson algebra L with $D(L_+)$ can now be obtained. Observe firstly, that we have the following quantum double intertwining relations between L_+ and L_+^0 :

$$[N, \nu] = 0, \quad (43)$$

$$[N, \beta] = -\beta, \quad (44)$$

$$[\nu, a^\dagger] = a^\dagger, \quad (45)$$

$$[\beta, a^\dagger] = \left[\frac{\nu + N + 1}{2} \right]_q - \left[\frac{\nu + N - 1}{2} \right]_q. \quad (46)$$

Secondly note that the element $\nu - N$ is central in $D(L_+)$ and generates a two-sided Hopf ideal, call it M . The quotient Hopf algebra $D(L_+)/M$, in which $\nu = N$ can be identified with the q -boson algebra L by identifying β with a and 1^* with I . The R -matrix is now given by

$$R = q^{(N - \frac{i\alpha}{\gamma}I) \otimes (N - \frac{i\alpha}{\gamma}I)} \sum_{k=0}^{\infty} \frac{i^k q^{-k(k+1)/4}}{\prod_{j=1}^k \left[\frac{j}{2} \right]_q} q^{kN/2} (a^\dagger)^k \otimes q^{-kN/2} a^k. \quad (47)$$

and satisfies the Yang-Baxter equation.

The differences between (19) and (47) can now be read off and the cause of the inconsistency of (19) is obvious. The representation theory of (7) in the spirit of [31], the relation of (47) with the R -matrix of $U_q(sl(2))$, its connection with representations of the braid group and possible relation with link invariances are under investigation. Finally we conclude with the observation that besides the Hopf algebra structure given by (12) a more general one exists given by

$$\begin{aligned} \Delta(N) &= N \otimes I + I \otimes N + \beta I \otimes I, \\ \Delta(a) &= \left(a \otimes q^{mN} \pm (-1)^K i q^{(m\pm 1)N} \otimes a \right) e^{i\pi(2K+1)m/2}, \\ \Delta(a^\dagger) &= \left(a^\dagger \otimes q^{-(m\pm 1)N} \pm (-1)^K i q^{-mN} \otimes a^\dagger \right) e^{i\pi(2K+1)(m\pm 1)/2}, \\ \varepsilon(N) &= -\beta, \quad \varepsilon(a) = \varepsilon(a^\dagger) = 0, \quad \varepsilon(I) = 1, \\ S(N) &= -N - 2\beta I, \\ S(a) &= \pm i (-1)^K q^{-(m\pm 1)N} a q^{mS(N)}, \\ S(a^\dagger) &= \pm i (-1)^K q^{mN} a^\dagger q^{-(m\pm 1)S(N)}. \end{aligned} \quad (48)$$

where $\beta = \frac{i\pi(2K+1)}{2\gamma}$, $\gamma = \ln q$, m is an integer or half-integer, and K is an integer. The corresponding R -matrix is now given by

$$R = q^{\mp(N+\beta I) \otimes (N+\beta I)} \sum_{k=0}^{\infty} \frac{(\pm i)^k (-1)^{Kk}}{\prod_{j=1}^k \left[\frac{j}{2} \right]_q} q^{-mk^2 \mp \frac{1}{4}k(k-1)} q^{mkN} (a^\dagger)^k \otimes q^{-mkN} a^k. \quad (49)$$

The choices $m = 1/2$, $K = -2\kappa - 1$ ($\kappa \in \mathbb{Z}$) and the lower sign in (48) and (49), lead directly to (12) and (47) respectively.

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